Exercise 3

In each of the Exercises 1 through 3, use residues to find the inverse Laplace transform $f(t)$ corresponding to the given function $F(s)$. Do this in a formal way, without full justification,

$$F(s) = \frac{12}{s^3 + 8}.$$

Suggestion: After finding the three cube roots -2 and $1 \pm \sqrt{3}i$ of -8, it is helpful to notice that the property $z + \bar{z} = 2\,\mathrm{Re}\,z$ of complex numbers enables one to write

$$\frac{e^{i\sqrt{3}t}}{-1 + i\sqrt{3}} + \frac{e^{-i\sqrt{3}t}}{-1 - i\sqrt{3}} = 2\,\mathrm{Re}\left[\frac{e^{i\sqrt{3}t}}{-1 + i\sqrt{3}}\right]$$

Ans. $f(t) = e^{-2t} + e^{t}(\sqrt{3}\sin\sqrt{3}t - \cos\sqrt{3}t)$.

Solution

Start off by finding the singularities of $F(s)$.

$$s^3 + 8 = 0 \quad \to \quad s^3 = -8$$

Use De Moivre's theorem to determine the roots.

$$\begin{aligned} s_{k+1} &= \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0 + 2\pi k}{n}\right)\right], \quad k = 0, 1, \ldots n-1 \\ &= \sqrt[3]{8}\exp\left[i\left(\frac{\pi + 2\pi k}{3}\right)\right], \quad k = 0, 1, 2 \end{aligned}$$

Hence, there are three singularities.

$$s_1 = 1 + \sqrt{3}i \quad s_2 = -2 \quad s_3 = 1 - \sqrt{3}i$$

The inverse Laplace transform is given by

$$f(t) = \sum_{n=1}^{3} \operatorname*{Res}_{s=s_n}[e^{st}F(s)].$$

We have

$$e^{st}F(s) = \frac{12e^{st}}{(s - s_1)(s - s_2)(s - s_3)}.$$

Since all the factors in the denominator have multiplicity 1, s_1, s_2, and s_3 are simple poles, so the residues are of the form $\phi_n(s_n)$.

Let $\phi_1(s) = \dfrac{12e^{st}}{(s - s_2)(s - s_3)}$. Then $\operatorname*{Res}_{s=s_1}[e^{st}F(s)] = \operatorname*{Res}_{s=s_1}\dfrac{\phi_1(s)}{s - s_1} = \phi_1(s_1) = -\dfrac{2i\sqrt{3}e^{(1+i\sqrt{3})t}}{3 + i\sqrt{3}}$.

Let $\phi_2(s) = \dfrac{12e^{st}}{(s - s_1)(s - s_3)}$. Then $\operatorname*{Res}_{s=s_2}[e^{st}F(s)] = \operatorname*{Res}_{s=s_2}\dfrac{\phi_2(s)}{s - s_2} = \phi_2(s_2) = \dfrac{12e^{-2t}}{(3 + i\sqrt{3})(3 - i\sqrt{3})}$.

Let $\phi_3(s) = \dfrac{12e^{st}}{(s - s_1)(s - s_2)}$. Then $\operatorname*{Res}_{s=s_3}[e^{st}F(s)] = \operatorname*{Res}_{s=s_3}\dfrac{\phi_3(s)}{s - s_3} = \phi_3(s_3) = \dfrac{2i\sqrt{3}e^{(1-i\sqrt{3})t}}{3 - i\sqrt{3}}$.

Summing the residues we obtain $f(t)$, the inverse Laplace transform of $F(s)$.

$$\begin{aligned}
f(t) = \sum_{n=1}^{3} \operatorname*{Res}_{s=s_n}[e^{st}F(s)] &= -\frac{2i\sqrt{3}e^{(1+i\sqrt{3})t}}{3+i\sqrt{3}} + \frac{12e^{-2t}}{(3+i\sqrt{3})(3-i\sqrt{3})} + \frac{2i\sqrt{3}e^{(1-i\sqrt{3})t}}{3-i\sqrt{3}} \\
&= -\frac{6(1+i\sqrt{3})e^{(1+i\sqrt{3})t}}{(3+i\sqrt{3})(3-i\sqrt{3})} + \frac{12e^{-2t}}{(3+i\sqrt{3})(3-i\sqrt{3})} + \frac{6(-1+i\sqrt{3})e^{(1-i\sqrt{3})t}}{(3+i\sqrt{3})(3-i\sqrt{3})} \\
&= \frac{-6(1+i\sqrt{3})e^{(1+i\sqrt{3})t} + 12e^{-2t} + 6(-1+i\sqrt{3})e^{(1-i\sqrt{3})t}}{(3+i\sqrt{3})(3-i\sqrt{3})} \\
&= \frac{-6(1+i\sqrt{3})e^{(1+i\sqrt{3})t} + 12e^{-2t} + 6(-1+i\sqrt{3})e^{(1-i\sqrt{3})t}}{12} \\
&= -\frac{1+i\sqrt{3}}{2}e^{(1+i\sqrt{3})t} + e^{-2t} + \frac{-1+i\sqrt{3}}{2}e^{(1-i\sqrt{3})t} \\
&= e^{-2t} + e^{t}\left(\frac{-1-i\sqrt{3}}{2}e^{i\sqrt{3}t} + \frac{-1+i\sqrt{3}}{2}e^{-i\sqrt{3}t}\right) \\
&= e^{-2t} + e^{t}\left(-\frac{e^{i\sqrt{3}t}+e^{-i\sqrt{3}t}}{2} + \sqrt{3}\cdot\frac{e^{i\sqrt{3}t}-e^{-i\sqrt{3}t}}{2i}\right)
\end{aligned}$$

Therefore,

$$f(t) = e^{-2t} + e^{t}(-\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t).$$